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Equivariant semialgebraic vector bundles

Myung-Jun Choi [a,1], Tomohiro Kawakami [b], Dae Heui Park [c,*]

[a] *Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon 305-701, South Korea*
[b] *Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan*
[c] *Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon 305-701, South Korea*

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Abstract

Let G be a compact semialgebraic group. We prove that any semialgebraic G-vector bundle over a semialgebraic G-set has a semialgebraic classifying G-map. Moreover we prove that the set of semialgebraic G-isomorphism classes of semialgebraic G-vector bundles over a semialgebraic G-set corresponds bijectively to the set of topological G-isomorphism classes of topological G-vector bundles over it. As an application of them, we prove that for any two semialgebraic G-maps f, h between semialgebraic G-sets M and N, if they are G-homotopic and ξ is a semialgebraic G-vector bundle over N, then $f^*(\xi)$ and $h^*(\xi)$ are semialgebraically G-isomorphic. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The semialgebraic category lies between the topological category and the algebraic category, and it is a useful tool to understand the relation between them. The semialgebraic category is less rigid than the algebraic category, and in some sense it is similar to the PL category.

* Corresponding author. The author wish to acknowledge the financial support of the Korea Research Foundation in the program year of 1999.
E-mail addresses: choimj@math.kaist.ac.kr (M.-J. Choi), kawa@center.wakayama-u.ac.jp (T. Kawakami), dhpark@math.kaist.ac.kr (D.H. Park).
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A central question in semialgebraic category is whether a given topological situation can be realized semialgebraically. Another question is to compare topological objects with the corresponding semialgebraic objects. In the present paper we treat the above questions for equivariant semialgebraic vector bundles. Note that non-equivariant semialgebraic vector bundles are introduced in [3].

In this paper, G denotes a compact semialgebraic group and a semialgebraic map between semialgebraic spaces is assumed to be continuous, unless otherwise stated.

Note that any compact semialgebraic group admits a unique compact Nash group structure [11], thus it admits a compact Lie group structure. Conversely, any compact Lie group admits a unique algebraic group structure (e.g., [12]), hence it admits a compact semialgebraic group structure.

A *semialgebraic G-set* means a G-invariant semialgebraic set in some orthogonal G-representation space, and we use a semialgebraic space as in the sense of [5]. Every semialgebraic set is a semialgebraic space in this sense.

Theorem 1.1. *Let G be a compact semialgebraic group. If $\xi = (E, p, M)$ is a semialgebraic G-vector bundle over a semialgebraic G-set M, then there exist a G-representation space Ω and a surjective semialgebraic G-morphism $f : \underline{\Omega} \to \xi$, where $\underline{\Omega}$ denotes the trivial G-vector bundle over M with fiber Ω.*

Theorem 1.1 implies that every semialgebraic G-vector bundle over a semialgebraic G-set has a semialgebraic classifying G-map. We can also view the total space of any semialgebraic G-vector bundle over a semialgebraic G-set as a semialgebraic G-set.

Let M be a semialgebraic G-set. Let $\mathrm{Vect}_G^{\mathrm{sem}}(M)$ (respectively $\mathrm{Vect}_G^{\mathrm{top}}(M)$) denote the set of semialgebraic (respectively topological) G-isomorphism classes of semialgebraic (respectively topological) G-vector bundles over M. Then there is a canonical map

$$\kappa : \mathrm{Vect}_G^{\mathrm{sem}}(M) \to \mathrm{Vect}_G^{\mathrm{top}}(M)$$

which sends the semialgebraic G-isomorphism class $[\xi]_{\mathrm{sem}}$ of a semialgebraic G-vector bundle ξ over M to the topological G-isomorphism class $[\xi]_{\mathrm{top}}$ of ξ.

Theorem 1.2. *Let G be a compact semialgebraic group and let M be a semialgebraic G-set. Then the canonical map κ is bijective.*

As an application of Theorem 1.2, we have the equivariant semialgebraic version of the homotopy property for semialgebraic G-vector bundles as follows.

Theorem 1.3. *Let G be a compact semialgebraic group and let $f, h : M \to N$ be semialgebraic G-maps between semialgebraic G-sets. If ξ is a semialgebraic G-vector bundle over N and f is G-homotopic to h, then the pull-back bundles $f^*(\xi)$ and $h^*(\xi)$ are semialgebraically G-isomorphic.*

We obtain the following as another application of Theorem 1.2.

Theorem 1.4. *Let G be a finite group and let ξ and η be strongly Nash G-vector bundles over an affine Nash G-manifold. If ξ and η are topologically G-isomorphic, then they are Nash G-isomorphic.*

In Theorem 1.4, we cannot drop the condition "strongly Nash" even if $G = 1$ [8].

2. Preliminaries on semialgebraic G-sets

A *semialgebraic set* is a subset of some $\mathbb{R}^n$ which is a finite union of sets of the form $\{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_k(x) = 0,\ h_1(x) > 0,\ \ldots,\ h_l(x) > 0\}$, where $f_i, h_j \in \mathbb{R}[x_1, \ldots, x_n]$. Note that the semialgebraic sets in $\mathbb{R}^n$ form the smallest family of subsets containing all sets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$, where $p \in \mathbb{R}[x_1, \ldots, x_n]$, such that it is closed under taking finite intersections, finite unions and complements. A continuous map $f : M \to N$ between semialgebraic sets is called a *semialgebraic map* if the graph of f is a semialgebraic subset of $M \times N$.

A *semialgebraic space* is an object obtained by pasting finitely many semialgebraic sets together along open semialgebraic subsets. We refer to [5,6] for the details of semialgebraic spaces.

A group G is a *semialgebraic group* if G is a semialgebraic set and that the group operations $G \times G \to G$ and $G \to G$ are semialgebraic maps. Typical examples are algebraic groups and subgroups of some $GL_n(\mathbb{R})$ which are semialgebraic sets. A group homomorphism between two semialgebraic groups is *semialgebraic* if it a semialgebraic map. Let G and G' be semialgebraic groups. A semialgebraic group homomorphism $\phi : G \to G'$ is called a *semialgebraic group isomorphism* if there exists a semialgebraic group homomorphism $\phi' : G' \to G$ such that $\phi \circ \phi' = \mathrm{id}$ and $\phi' \circ \phi = \mathrm{id}$.

A *representation map* (respectively an *orthogonal representation map*) of G is a semialgebraic group homomorphism from G to some $GL_n(\mathbb{R})$ (respectively some $O_n(\mathbb{R})$). Two representation maps $\phi, \psi : G \to GL_n(\mathbb{R})$ are *equivalent* if there exists an $A \in GL_n(\mathbb{R})$ such that $\psi(g) = A^{-1}\phi(g)A$ for all $g \in G$. In this case, we say that the representation spaces of ϕ, ψ are *equivalent*.

Proposition 2.1. *If G is a compact semialgebraic group, then every G-representation space is equivalent to an orthogonal G-representation space.*

Proof. Let Ω be the representation space of a representation map $\phi : G \to GL_n(\mathbb{R})$. Then $\phi(G)$ is a compact semialgebraic subgroup of $GL_n(\mathbb{R})$. Since the orbit map $\pi : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})/\phi(G)$ is a polynomial map and $\phi(G) = \pi^{-1}(\pi(e))$, $\phi(G)$ is a compact algebraic group, where e denotes the identity element of $GL_n(\mathbb{R})$. Replacing G by $\phi(G)$, we may assume that Ω is an algebraic G-representation space at the beginning. Using the averaging operator A constructed from the Haar measure, we have a G-invariant inner product of Ω

because $A(f)$ is a polynomial map if so is f [7, 4.1]. Therefore Ω is equivalent to an orthogonal G-representation. $\square$

By Proposition 2.1, we may assume that all G-representation spaces are orthogonal.

Definition 2.2.
(1) A *semialgebraic G-space* is a pair (X, θ) consisting of a semialgebraic space X and a group action $\theta : G \times X \to X$ which is semialgebraic. For simplicity of notation, we write X instead of (X, θ).
(2) Let X and Y be semialgebraic G-spaces. A semialgebraic map $f : X \to Y$ is called a *semialgebraic G-map* if it is a G-map. We say that X and Y are *semialgebraically G-homeomorphic* if there exist semialgebraic G-maps $h : X \to Y$ and $k : Y \to X$ such that $h \circ k = \mathrm{id}$ and $k \circ h = \mathrm{id}$.
(3) A *semialgebraic G-set* means a G-invariant semialgebraic set in some G-representation space.
(4) A semialgebraic G-space is said to be *affine* if it is semialgebraically G-homeomorphic to some semialgebraic G-set in a G-representation space.

Typical examples of semialgebraic G-sets are algebraic G-sets and affine $\mathcal{C}^r$ $(r \geqslant 0)$ Nash G-manifolds.

The following is the equivariant semialgebraic version of Urysohn's lemma. This is the equivariant version of [6, 1.6].

Lemma 2.3 [13, 2.10]. *Let M be a semialgebraic G-set and let A and B be disjoint closed semialgebraic G-subsets of M. Then there exists a G-invariant semialgebraic function $f : M \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

A *finite open semialgebraic G-CW complex* is a G invariant semialgebraic subspace of a finite semialgebraic G-CW complex X obtained by removing some finite open G-cells of X. Here a finite G-CW complex is called *semialgebraic* if every G-cell is a semialgebraic G-set and the corresponding characteristic maps are semialgebraic. The standard G-CW complexes are called here *complete G-CW* complexes.

Proposition 2.4 [13]. *Let (M, N) be a pair of semialgebraic G-sets such that N is closed in M. Then there exist a finite open semialgebraic G-CW complex X and its finite open semialgebraic G-CW subcomplex A such that (M, N) is semialgebraically G-homeomorphic to (X, A).*

Proposition 2.5 [13]. *Let M be a semialgebraic G-set. Then there exist a compact semialgebraic G-subset A of M and a semialgebraic G-strong deformation retraction $R : M \times [0, 1] \to M$ such that $R_0 := R(\cdot, 0) = \mathrm{id}_M$, $R(a, t) = a$ for all $(a, t) \in A \times [0, 1]$ and $r := R(\cdot, 1) : M \to A$ is a semialgebraic G-retraction.*

Proof. It was already proved in [13]. But we sketch the proof because we will use the construction of r in the proof of Proposition 4.9. Let Ω be a G-representation space containing M as a semialgebraic G-set. Then, by Proposition 2.4, M admits a semialgebraic G-CW complex structure X with $|X| = M$. We replace X by its barycentric subdivision X'. For the semialgebraic characteristic G-map $Gf_c : G/H \times \mathrm{int}(D^n) \to c$ of an open G-cell c of X, let σ denote $Gf_c(eH \times \mathrm{int}(D^n))$ and let $G\sigma$ (respectively Gf_σ) denote c (respectively Gf_c). Then for each open G-n-cell $G\sigma$, there exists a semialgebraic characteristic G-map $Gf_\sigma : G/H \times \delta \to G\overline{\sigma} \subset X$, where δ is a subset of a compact standard n-simplex Δ^n by removing some finite open lower-dimensional faces of Δ^n and $\overline{\sigma}$ denotes the closure of σ in X. Thus $\sigma = Gf_\sigma(eH \times \mathrm{int}(\delta))$ and $\overline{\sigma} = Gf_\sigma(eH \times \delta)$.

Let A denote the maximum complete (and thus compact) G-CW subcomplex of X. In other words, A is the union of all open G-cells $G\sigma$ of X such that $\mathrm{cl}(G\sigma) \subset M$, where $\mathrm{cl}(G\sigma)$ denotes the closure of $G\sigma$ in Ω. Namely, A is the union of open G-cells which have semialgebraic characteristic G-maps $Gf_\sigma : G/H \times \delta \to G\overline{\sigma} = \overline{G\sigma}$ such that δ is some compact standard n-simplex Δ^n. Then $G\overline{\sigma} \cap A \neq \emptyset$ for all open G-cells $G\sigma$ of X. Moreover the star $\mathrm{St}_X(A)$ of A in X is X.

Let $\mathcal{C}_n$ be the set of open G-n-cells of X such that $G\sigma \cap A = \emptyset$. Clearly $\mathcal{C}_n$ is a finite set and $\mathcal{C}_0 = \emptyset$.

Let $X_0 = A$ and $X_n = A \cup X^{(n)}$ for $n \geqslant 1$, where $X^{(n)}$ is the n-skeleton of M. Clearly $X_n = A \cup \{G\sigma \mid G\sigma \in \mathcal{C}_k,\ 0 \leqslant k \leqslant n\}$.

For each open G-n-cell $G\sigma \in \mathcal{C}_n$, there exists a nonequivariant semialgebraic strong deformation retraction $F^n_{\overline{\sigma}} : \overline{\sigma} \times [0,1] \to \overline{\sigma}$ from $\overline{\sigma}$ to $\partial\sigma = \overline{\sigma} - \mathrm{int}(\sigma)$. Hence we have a semialgebraic strong G-deformation retraction

$$R^n_{G\overline{\sigma}} : G\overline{\sigma} \times [0,1] \to G\overline{\sigma} \quad (gx, t) \mapsto gF^n_{\overline{\sigma}}(x, t)$$

from $G\overline{\sigma}$ to $G\partial\sigma (\subset X_{n-1})$.

Put $R^n = \bigcup\{R^n_{G\overline{\sigma}} \mid G\sigma \in \mathcal{C}_n\}$. Then $R^n : X_n \times [0,1] \to X_n$ is a semialgebraic strong G-deformation retraction from X_n to X_{n-1}. We denote $R^n(\cdot, 1)$ and $R^n_{G\overline{\sigma}}(\cdot, 1)$ by r_n and $r^n_{G\overline{\sigma}}$, respectively. Clearly $r_n = \bigcup\{r^n_{G\overline{\sigma}} \mid G\sigma \in \mathcal{C}_n\}$.

Then we can define $R^{i+1} \circledast R^i : X_{i+1} \times [0,1] \to X$ for $1 \leqslant i$,

$$R^{i+1} \circledast R^i(x,t) = \begin{cases} R^{i+1}(x, 2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ R^i\big(R^{i+1}(x,1), 2t-1\big) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

The map R is obtained by

$$R = R^m \circledast R^{m-1} \circledast \cdots \circledast R^2 \circledast R^1 : X \times [0,1] \to X$$

which is the desired semialgebraic strong G-deformation retraction from X to A, where $m = \min\{n \in \mathbb{N} \mid X = X_n\}$. In particular, $r = R_1 = r_1 \circ r_2 \circ \cdots \circ r_{m-1} \circ r_m$ is a semialgebraic G-retraction from X to A. $\square$

Note that r is a semialgebraic G-homotopy equivalence because the inclusion $i : A \hookrightarrow M$ is a semialgebraic G-homotopy inverse.

The following proposition implies that every topological G-homotopy class is represented by a semialgebraic G-map.

Proposition 2.6. *Let M and N be semialgebraic G-sets. Then every continuous G-map $f : M \to N$ is G-homotopic to a semialgebraic G-map $h : M \to N$.*

Proof. First of all, we prove this proposition when M is compact.

Let N be a semialgebraic G-set in a G-representation space Ξ. Let $R_N : N \times [0,1] \to N$ be a semialgebraic G-strong deformation retraction from N to a compact semialgebraic G-subset B of N with $r_N = R_N(\cdot, 1)$ as in Proposition 2.5. Put $K : M \times [0,1] \to N$, $K(x,t) = R_N(f(x), t)$. Then K is a G-homotopy from f to $r_N \circ f$.

We now construct a semialgebraic G-map which is G-homotopic to $r_N \circ f$. Since B is compact, there exists a real number $r > 0$ such that B is contained in the interior of $D := \{x \in \Xi \mid \|x\| \leqslant r\}$. By Proposition 2.4, we may assume that (D, B) is a pair of finite compact complete semialgebraic G-*CW* complexes. Put $V = \mathrm{St}_D(B)$ and let r_V be a semialgebraic G-retraction from V to B. We approximate $r_N \circ f$ by a polynomial map $p : M \to \Xi$. By [7, 4.1], after averaging p, we may assume that p is equivariant. Since V is an open subset of Ξ, we can take p with $p(M) \subset V$ if p is sufficiently close to $r_N \circ f$. We may assume the line segment $(1-t)r_N \circ f(x) + tp(x)$, $0 \leqslant t \leqslant 1$, lies in V. Then $h := r_V \circ p : M \to B$ is a semialgebraic G-approximation of $r_N \circ f$.

Now we consider the map $P : M \times [0,1] \to B$, $P(x,t) = r_V((1-t)r_N \circ f(x) + tp(x))$ is a G-homotopy from $r_N \circ f$ to h. Therefore, the homotopy composition $K * P : M \times [0,1] \to N$,

$$K * (x,t) = \begin{cases} K(x, 2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ P(x, 2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

is a G-homotopy from f to h, and the proposition follows.

Now we prove this proposition when M is general. Let $R_M : M \times [0,1] \to M$ be a semialgebraic G-strong deformation retraction from M to a compact semialgebraic G-subset A of M with $r_M = R_M(\cdot, 1)$ as in Proposition 2.5. By the compact case, there exist a G-homotopy $F : A \times [0,1] \to N$ and a semialgebraic G-map $u : A \to N$ such that $F(x,0) = f(x)$, $F(x,1) = u(x)$ for all $x \in A$. Put $H = F \circ (r_M \times \mathrm{id}_{[0,1]}) : M \times [0,1] \to N$. Then H is a G-homotopy from $f \circ r_M$ to $u \circ r_M$. Note that

$$f = f \circ \mathrm{id}_M \underset{f \circ R_M}{\sim} f \circ r_M \underset{H}{\sim} u \circ r_M.$$

Therefore f is G-homotopic to a semialgebraic G-map $h := u \circ r_M$. □

3. Semialgebraic *G*-vector bundles

Recall the definition of semialgebraic vector bundles [3].

Definition 3.1.

(1) Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Let $\xi = (E, p, M)$ be a vector bundle of rank k over M. A family of local trivializations $(U_i, \varphi_i : U_i \times \mathbb{R}^k \to p^{-1}(U_i))_{i \in I}$ of ξ

is said to be *a semialgebraic atlas* of ξ if $(U_i)_{i\in I}$ is a finite open semialgebraic covering of M and the map $\varphi_i^{-1}\circ\varphi_j|_{(U_i\cap U_j)\times\mathbb{R}^k} : (U_i\cap U_j)\times\mathbb{R}^k \to (U_i\cap U_j)\times\mathbb{R}^k$ is semialgebraic, for every pair $(i, j)\in I\times I$. Two semialgebraic atlases are *equivalent* if their union is still a semialgebraic atlas. *A semialgebraic vector bundle* is a vector bundle $\xi=(E, p, M)$ equipped with an equivalence class of semialgebraic atlases. Note that the total space of a semialgebraic vector bundle is a semialgebraic space.

(2) Let $(\xi, (U_i, \varphi_i)_{i\in I})$ and $(\xi', (U'_j, \varphi'_j)_{j\in J})$ be two semialgebraic vector bundles over M. A morphism $\psi : \xi\to\xi'$ of vector bundles is said to be a *semialgebraic morphism* if the map

$$\left(\varphi'_j\right)^{-1}\circ\psi\circ\varphi_i|_{(U_i\cap U'_j)\times\mathbb{R}^k} : \left(U_i\cap U'_j\right)\times\mathbb{R}^k \to \left(U_i\cap U'_j\right)\times\mathbb{R}^k$$

is semialgebraic, for every $(i, j)\in I\times J$. A continuous section s of ξ is said to be a *semialgebraic section* if the map $\varphi_i^{-1}\circ s|_{U_i} : U_i\to U_i\times\mathbb{R}^k$ is semialgebraic, for every $i\in I$.

By abuse of notion, we denote by $\xi=(E, p, M)$ a semialgebraic vector bundle without specifying the atlas defining its structure.

Definition 3.2. Let G be a semialgebraic group. A G-vector bundle $\xi=(E, p, M)$ is a *semialgebraic G-vector bundle* if ξ satisfies the following three conditions:

(1) The total space E is a semialgebraic G-space and the base space M is a semialgebraic G-set.
(2) The projection $p : E\to M$ is a semialgebraic G-map, and g sends $E|_x$ to $E|_{gx}$ linearly for all $g\in G$.
(3) (E, p, M) is a semialgebraic vector bundle when we forget the action.

For a G-representation Ω, let $\underline{\Omega}$ denote the trivial G-vector bundle $M\times\Omega\to M$.

A *semialgebraic G-vector bundle morphism* (briefly, *semialgebraic G-morphism*) $\varphi : \xi\to\xi'$ between two semialgebraic G-vector bundles $\xi=(E, p, M)$ and $\xi'=(E', p', M)$ is a semialgebraic G-map $\varphi : E\to E'$ such that $p'\circ\varphi=p$ and φ is linear on each fiber. A semialgebraic G-morphism $\varphi : \xi\to\xi'$ is called a *semialgebraic G-vector bundle isomorphism* (briefly, *semialgebraic G-isomorphism*) if there exists a semialgebraic G-morphism $\varphi' : \xi'\to\xi$ such that $\varphi\circ\varphi'=\mathrm{id}$ and $\varphi'\circ\varphi=\mathrm{id}$. For two semialgebraic (respectively topological) G-vector bundles ξ and ξ', we sometimes simply write $\xi\cong^{\mathrm{sem}}_G\xi'$ (respectively $\xi\cong^{\mathrm{top}}_G\xi'$) to mean they are semialgebraically G-isomorphic (respectively topologically G-isomorphic).

A semialgebraic section s is called a *semialgebraic G-section* if it is G-equivariant.

By a way similar to [8, 3.1], we have the following proposition.

Proposition 3.3. *Let G be a semialgebraic group. If ξ and ξ' are two semialgebraic G-vector bundles over a semialgebraic G-set M, then $\xi\oplus\xi'$, $\xi\otimes\xi'$,* $\mathrm{Hom}(\xi, \xi')$ *and the dual bundle $\xi^\vee$ are semialgebraic G-vector bundles over M.*

Let Ω be a G-representation space with underlying vector space $\mathbb{R}^n$. The set $\mathrm{End}(\Omega)$ of endomorphisms of Ω is a G-representation space with the action given by $(g, L) \to gLg^{-1}$ for $g \in G$ and $L \in \mathrm{End}(\Omega)$.

For a natural number k, let

$$\begin{aligned} G(\Omega, k) &= \{L \in \mathrm{End}(\Omega) \mid L^2 = L,\ L^t = L,\ \mathrm{trace}(L) = k\}, \\ E(\Omega, k) &= \{(L, u) \in \mathrm{End}(\Omega) \times \Omega \mid L \in \mathbb{G}(\Omega, k),\ Lu = u\}, \\ E^{\perp}(\Omega, k) &= \{(L, u) \in \mathrm{End}(\Omega) \times \Omega \mid L \in \mathbb{G}(\Omega, k),\ Lu = 0\}. \end{aligned}$$

Here L^t denotes the adjoint of L. If one chooses the standard basis of Ω, then $\mathrm{End}(\Omega)$ is canonically identified with the set $M_n(\mathbb{R})$ $(= \mathbb{R}^{n^2})$ of $n \times n$ matrices, and L^t is obtained by transposing L. This description specifies $G(\Omega, k)$ and $E(\Xi, k)$ as nonsingular algebraic G-sets, and thus they are semialgebraic G-sets.

Define p (respectively $p^{\perp}$) by the canonical projection of $E(\Omega, k)$ (respectively $E^{\perp}(\Omega, k)$) over $G(\Omega, k)$. This defines G-vector bundle $\gamma(\Omega, k) = (E(\Omega, k), p, G(\Omega, k))$ (respectively $\gamma^{\perp}(\Omega, k) = (E(\Omega, k)^{\perp}, p^{\perp}, G(\Omega, k))$). The bundle $\gamma(\Omega, k)$ is called the *universal bundle* over $G(\Omega, k)$. Note that $\gamma(\Omega, k)$ is a strongly algebraic G-vector bundle, and thus a (strongly) semialgebraic G-vector bundle.

Definition 3.4. A semialgebraic G-vector bundle (E, p, M) is called *strongly semialgebraic* if there exist non-equivariant semialgebraic global sections $s_1, \ldots, s_n : M \to E$ such that:

(1) The vectors $s_1(x), \ldots, s_n(x)$ generate the fiber $E|_x = p^{-1}(x)$ for all $x \in M$.

(2) The sections $s_1, \ldots, s_n$ generate a finite dimensional G-invariant vector subspace of $\Gamma(E)$, where $\Gamma(E)$ denotes the set of all continuous global sections of E with the natural G-action, namely $(g \cdot s)(x) = g(s(g^{-1}(x)))$ for all $g \in G$ and $x \in M$.

The following theorem is the main result of this section.

Theorem 3.5. *Let G be a compact semialgebraic group. Then every semialgebraic G-vector bundle over a semialgebraic G-set is strongly semialgebraic.*

The next theorem states equivalent properties of strong semialgebraicity of semialgebraic G-vector bundles.

Theorem 3.6. *Let G be a compact semialgebraic group and let ξ be a semialgebraic G-vector bundle of rank k over a semialgebraic G-set M. Then the following five properties are equivalent.*

(1) *ξ is strongly semialgebraic.*

(2) *There exists a surjective semialgebraic G-morphism from a trivial G-vector bundle $\underline{\Omega}$ onto ξ for some G-representation space Ω.*

(3) *There exists an injective semialgebraic G-morphism from ξ to a trivial G-vector bundle $\underline{\Omega}$ for some G-representation space Ω.*

(4) *There exist a G-representation Ω and a semialgebraic G-map $f : M \to G(\Omega, k)$ such that ξ is semialgebraically G-isomorphic to $f^*(\gamma(\Omega, k))$.*
(5) *There exists a semialgebraic vector bundle η over M such that $\xi \oplus \eta$ is semialgebraically G-isomorphic to a trivial G-vector bundle $\underline{\Omega}$ for some G-representation space Ω.*

Proof. (3)⇒(4) Let $\varphi : \xi \to \underline{\Omega} = M \times \Omega$ be an injective semialgebraic G-morphism. We define $f : M \to G(\Omega, k)$ by $\{x\} \times f(x) = \varphi(p^{-1}(x))$. Then f is a semialgebraic G-map such that $\xi \cong_G^{\mathrm{sem}} f^*(\gamma(\Omega, k))$.

(4) ⇒ (5) Since $G(\Omega, k) \times \Omega = E(\Omega, k) \oplus E^{\perp}(\Omega, k)$, we have $f^*(\gamma(\Omega, k)) \oplus f^*(\gamma^{\perp}(\Omega, k)) \cong_G^{\mathrm{sem}} f^*(G(\Omega, k) \times \Omega)$. Thus $f^*(G(\Omega, k) \times \Omega)$ is trivial because so is $G(\Omega, k) \times \Omega$.

(5) ⇒ (2) It is trivial.

(2) ⇒ (3) Given a surjective semialgebraic G-morphism $\underline{\Omega} = M \times \Omega \to \xi$, we obtain, by dualizing, an injective semialgebraic G-morphism $\xi^{\vee} \to \underline{\Omega}^{\vee} \cong_G^{\mathrm{sem}} \underline{\Omega}$. By the preceding arguments imply that $\xi^{\vee}$ is a semialgebraic direct factor of the trivial bundle $\underline{\Omega}$. Hence ξ is a semialgebraic direct factor of $\underline{\Omega}$.

(1) ⇒ (2) There exist non-equivariant semialgebraic global sections $s_1, \ldots, s_n$ such that they generate a finite dimensional G-invariant vector subspace Ω of $\Gamma(E)$ and that for any $x \in M$, $s_1(x), \ldots, s_n(x)$ generates the fiber over x. Then the evaluation map $\varphi : \underline{\Omega} \to E$ defined by $\varphi(x, s) = s(x)$ is a surjective semialgebraic G-morphism.

(4) ⇒ (1) Take semialgebraic sections $s_1(x), \ldots, s_n(x)$ of $\underline{\Omega} = M \times \Omega$ generating each fiber, where $n = \dim \Omega$. Then $f(x)s_1(x), \ldots, f(x)s_n(x)$ are the desired family of semialgebraic sections. □

To prove Theorem 3.5, we need the following preparations.

Let X be an arbitrary set (not necessarily semialgebraic). Let K be an arbitrary field and let K^X denote a K-vector space of all functions from X to K.

Lemma 3.7. *If $\phi_1, \ldots, \phi_n \in K^X$ is a family of linearly independent vectors, then there exist $a_1, \ldots, a_n \in X$ such that the matrix $(\phi_j(a_i))_{1 \leqslant i,\ j \leqslant n}$ is a nonsingular matrix.*

Proof. By induction on n, suppose that we have found $a_1, \ldots, a_{n-1}$ such that $(\phi_j(a_i))_{1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n-1}$ is nonsingular. Consider

$$A(x) = \begin{pmatrix} & \phi_n(a_1) \\ (\phi_j(a_i))_{1 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n-1} & \vdots \\ & \phi_n(a_{n-1}) \\ \phi_1(x) \cdots \phi_{n-1}(x) & \phi_n(x) \end{pmatrix}.$$

Then we have

$$\det A(x) = \alpha_1 \phi_1(x) + \cdots + \alpha_n \phi_n(x), \quad \text{where } \alpha_n \neq 0.$$

If $\det A(x) = 0$ for all $x \in X$, it implies that $\{\phi_i\}_{1 \leqslant i \leqslant n}$ is linearly dependent, hence there exists a point $a_n \in X$ such that $\det A(a_n) \neq 0$. Thus we have found suitable $a_1, \ldots, a_n$. □

A semialgebraic G-vector bundle is called *weakly trivial* if it is non-equivariantly semialgebraically isomorphic to some trivial vector bundle.

Proposition 3.8. *Let (E, p, B) be a semialgebraic G-vector bundle over a connected semialgebraic G-set B with the trivial G-action. If (E, p, B) is weakly trivial, then (E, p, B) is strongly semialgebraic.*

Proof. Let $\{s_1, \dots, s_n\}$ be a family of non-equivariant global semialgebraic sections which is linearly independent at each fiber and $s_i(x)$'s generate E_x. Let $A(g, x) = (a_{i,j}(g, x))$ be an $n \times n$ matrix whose (i, j) components are defined by

$$gs_i(x) = \sum_j a_{i,j}(g, x)s_j(x).$$

Note that every fiber has an isomorphic G-representation because B is connected. Let V_x be the vector space of functions G to $\mathbb{R}$ spanned by (i, j) components of $A(*, x)$ for fixed x. Then all V_x are equal. More precisely, for all $x, y \in B$, $V_x = V_y$ in the set $\mathcal{C}(G, \mathbb{R})$ of continuous functions from G to $\mathbb{R}$.

Let $V := V_x \subset \mathcal{C}(G, \mathbb{R})$. Choose $\phi_1, \dots, \phi_m : G \to \mathbb{R}$ as a basis of V. Then we can uniquely write $a_{ij}(g, x) = \sum_k f_{ij}^k(x)\phi_k(g)$, where f_{ij}^k are functions defined on B. By Lemma 3.7, we can find $g_l \in G$ $(1 \leqslant l \leqslant m)$, such that $(\phi_k(g_l))$ is a nonsingular matrix. Thus f_{ij}^k can be written as a linear sum of $a_{ij}(g_l, x)$, which implies that f_{ij}^k are semialgebraic.

Now the next step is to approximate s_i by a semialgebraic section whose orbit generate a finite dimensional G-invariant subspace of $\Gamma(E)$. For any continuous function φ defined on G, and for each $s_i \in \Gamma(E)$, recall that the convolution $\varphi * s_i$ (e.g., [4, p. 140]) is defined by

$$\varphi * s_i(x) = \int_G \varphi(h)h \cdot s_i(x)\, \mathrm{d}h = \int_G \varphi(h)hs_i(x)\, \mathrm{d}h.$$

Then

$$\begin{aligned} \varphi * s_i(x) &= \int_G \varphi(h)\Bigl(\sum_j a_{ij}(h, x)s_j(x)\Bigr)\mathrm{d}h \\ &= \int_G \varphi(h)\Bigl(\sum_j\Bigl(\sum_k f_{ij}^k(x)\phi_k(h)\Bigr)s_j(x)\Bigr)\mathrm{d}h \\ &= \sum_{j,k}\Bigl\{f_{ij}^k(x)s_j(x)\int_G \varphi(h)\phi_k(h)\, \mathrm{d}h\Bigr\}. \end{aligned}$$

The convolution $\varphi * s_i$ is semialgebraic, since the last integral is just a constant in $\mathbb{R}$. Since $g(\varphi * s) = (g \cdot \varphi) * s = (\varphi g^{-1}) * s$, if we take φ as a representative function, then the convolution is contained in a finite-dimensional G-subspace of $\Gamma(E)$ (e.g., [4, III.5.6]). Thus it remains to take an appropriate function φ. Let U be a small open neighborhood of $e \in G$ such that $U = U^{-1}$. As in the proof of [4, III.3.1], let $\varphi' : G \to [0, \infty)$ be a continuous "bump function" with support contained in U such that $\varphi'(g) = \varphi'(g^{-1})$ and $\int_G \varphi' = 1$. It is well known that the convolution $\varphi' * s_i$ is close to the s if we take U small enough. By the Peter–Weyl theorem (e.g., [4, III.3.1]), we can choose a representative function φ close enough to φ'. Hence we have found a semialgebraic section close enough to s_i by $\varphi * s_i$. □

Note that we have shown the following fact in the proof of Proposition 3.8. Under the same assumption of Proposition 3.8, if $s : B \to E$ is a semialgebraic section, then the integral of s with respect to an arbitrary measure defined on G is again semialgebraic.

If $\xi = (E, p, B)$ is a strongly semialgebraic G-vector bundle over a semialgebraic G-set B, then there is a trivial G-vector bundle $\underline{W}$ and a surjective semialgebraic G-morphism $\varphi : \underline{W} \to \xi$. If B has the trivial G-action, then there exists a global semialgebraic section $\tilde{\varphi} : B \to \mathrm{Hom}(\underline{W}, \xi)$ such that:

(i) $\tilde{\varphi}(x) \in \mathrm{Hom}^G(W, E_x)$ for all $x \in B$.

(ii) The linear map $\tilde{\varphi}(x) : W \to E_x$ is surjective for all $x \in B$.

Conversely, if such a section exists, then ξ is strongly semialgebraic.

Proposition 3.9. *Let $\xi = (E, p, B)$ be a semialgebraic G-vector bundle over a semialgebraic set B with the trivial G-action. Let A be a closed semialgebraic subset of B such that $\xi|_A$ is strongly semialgebraic. If A admits a semialgebraic G-neighborhood retract, then $\xi|_V$ is a strongly semialgebraic G-vector bundle for some open semialgebraic neighborhood V of A in B.*

Proof. Since $\xi|_A$ is a strongly semialgebraic G-vector bundle, we have a semialgebraic section $\tilde{\varphi} : A \to \mathrm{Hom}(\underline{W}|_A, \xi|_A)$ which satisfies conditions (i) and (ii). Take a semialgebraic G-neighborhood retract U of A. Then we can extend $\tilde{\varphi}$ to a semialgebraic section on U which may not satisfy conditions (i) or (ii). We again write the extended section as $\tilde{\varphi}$. Now we consider its convolution with a constant function 1, namely $\tilde{\varphi}'(x) = \int g\tilde{\varphi}(x)\,\mathrm{d}g$. Then $\tilde{\varphi}'(x)$ lies in $\mathrm{Hom}^G(W, E_x)$ for every $x \in U$ and $\tilde{\varphi}'|_A = \tilde{\varphi}|_A$. Thus we get an extension of $\tilde{\varphi}$ which satisfies condition (i). Now the surjectivity is an open condition, hence we can find an open semialgebraic subset V with $A \subset V \subset U$ which satisfies condition (ii). □

Keep the notations as in Proposition 3.9. Let B be a semialgebraic G-set with the trivial G-action. Since A and $B - U$ are disjoint closed semialgebraic subsets of B, there exist a semialgebraic function $f : B \to [0, 1]$ such that $f^{-1}(0) = B - U$ and $f^{-1}(1) = A$ by Lemma 2.3. Thus we can extend a semialgebraic section s defined on U to be a global semialgebraic section $\tilde{s}$, more precisely $\tilde{s}(x) = f(x)s(x)$ such that $\tilde{s}|_A = s|_A$ and $\tilde{s}$ vanishes outside U. Moreover, if $s_1, \ldots, s_k$ are semialgebraic sections which generate a finite dimensional G-invariant vector subspace of the set $\Gamma(U, E)$ of continuous sections on U of E, then the global semialgebraic sections $\tilde{s}_1, \ldots, \tilde{s}_k$ generate a finite dimensional G-invariant vector subspace of $\Gamma(E)$.

Proof of Theorem 3.5. Let $\xi = (E, p, B)$ be a semialgebraic G-vector bundle over a semialgebraic G-set B, and we do not assume that B has the trivial G-action.

By Proposition 2.4, we may assume that B is a finite open semi-algebraic G-CW complex. Let $B^{(n)}$ denote the n-skeleton of B. By induction on n, suppose that $\xi|_{B^{(n-1)}}$ is a strongly semialgebraic G-vector bundle. Let $G\sigma$ be an open G-n-cell of B and $Gf_\sigma : G/H \times \delta \to G\overline{\sigma}$ the semialgebraic characteristic G-map of $G\overline{\sigma}$, where $\overline{\sigma}$ is the closure of σ in B and δ is a subset of a compact standard n-simplex Δ^n obtained by removing some finite open lower dimensional faces of Δ^n.

Consider the pull-back bundle $\xi' = (Gf_\sigma)^*\xi$ and let $\xi'' = \xi'|_{\{eH\}\times\delta}$ $(= (f_\sigma)^*\xi)$. Then ξ'' is an H-vector bundle over δ, where H acts trivially on δ. Since H acts trivially on δ, we can apply all previous arguments to this case.

By the inductive hypothesis, we know that $\xi''|_{\partial\delta}$ is a strongly semialgebraic H-vector bundle, where $\partial\delta = \delta - \operatorname{int}(\delta)$. Thus there exist semialgebraic sections $s_1, \dots, s_k$ on $\partial\delta$ which form a basis of a finite dimensional H-invariant vector space in $\Gamma(\xi''|_{\partial\delta})$ and generate at every fiber.

By Proposition 3.9, we can find extensions $\tilde{s}_1, \dots, \tilde{s}_k$ on δ such that they form a basis of a finite dimensional H-invariant vector space. But they may not satisfy the condition (ii). Hence we need more sections.

Since $\partial\delta$ is a closed semialgebraic subset of δ, there exists an open semialgebraic neighborhood U of $\partial\delta$ in δ such that $\tilde{s}_i|_U$ satisfies the condition (ii). We know by Proposition 3.8, on δ, there are semialgebraic sections $s_{k+1}, \dots, s_l$ which satisfy conditions (i) and (ii).

Because $\partial\delta$ and $\delta - U$ are disjoint closed semialgebraic subsets of δ, by Lemma 2.2, there exists a semialgebraic function $f : \delta \to [0, 1]$ such that $f^{-1}(0) = \partial\delta$ and $f^{-1}(1) = \delta - U$.

Let $\tilde{s}_i(x) = f(x)s_i(x)$, $k + 1 \leqslant i \leqslant l$. Then they are semialgebraic sections such that $\tilde{s}_j|_{\delta-U} = s_j|_{\delta-U}$ and vanish on $\partial\delta$ for $k + 1 \leqslant j \leqslant l$. Then $\tilde{s}_1, \dots, \tilde{s}_k, \tilde{s}_{k+1}, \dots, \tilde{s}_l$ satisfy the condition (i) and (ii). Thus we get a surjective semialgebraic H-morphism $\Phi : \underline{V_1} \oplus \underline{V_2} \to \xi''$ such that $\Phi|_{\partial\delta\times V_2} = 0$, where $V_1 = \langle \tilde{s}_1, \dots, \tilde{s}_k\rangle$ (respectively $V_2 = \langle \tilde{s}_{k+1}, \dots, \tilde{s}_l\rangle$) denotes the finite dimensional H-invariant vector subspace of $\Gamma(\xi'')$ generated by $\tilde{s}_1, \dots, \tilde{s}_k$ (respectively $\tilde{s}_{k+1}, \dots, \tilde{s}_l$).

Let $V = V_1 \oplus V_2$. We claim that $G \times_H D$ is semialgebraically G-imbeddable into some G-representation space W, where D denotes the closed unit ball of V. By [11], every compact semialgebraic group is semialgebraically group isomorphic to a compact Nash group. Thus we may assume that G is a Nash G-submanifold of some G-representation space W' by [9, 2.7]. Hence $G \times V$ is an Nash H-submanifold of $W' \times V$, and $G \times_H V$ is a Nash G-manifold [9, 4.5]. Hence by the Nash slice theorem [9, 4.1] and by the equivariant semialgebraic partition of unity [13, 2.9], we can find a semialgebraic G-imbedding i from $G \times_H D$ to a G-representation space W because $G \times_H D$ is compact. Moreover we may assume that $i(\tilde{s}_1), \dots, i(\tilde{s}_l)$ are linearly independent vectors of W.

Using this imbedding, we have a new surjective semialgebraic H-morphism $\Psi : \underline{W} = \underline{V_1'} \oplus \underline{V_2'} \oplus \underline{V_3'} \to \xi''$ such that $\Psi|_{\partial\delta\times V_2'\times V_3'} = 0$ and $\Psi(v_1, v_2, v_3) = \Psi(v_1, v_2, 0)$ for all $v = (v_1, v_2, v_3) \in W$, where V_1' (respectively V_2') denotes the linear subspace generated by

$i(\tilde{s}_1), \dots, i(\tilde{s}_k)$ (respectively $i(\tilde{s}_{k+1}), \dots, i(\tilde{s}_l)$), and V_3' denotes the orthogonal complement of $V_1' \oplus V_2'$ in W. Note that V_1, V_2, V_3 are H-invariant.

On the other hand, if X is a G-space, then $G \times_H X \cong G/H \times X$. Taking $X = W$, the vector bundle $G \times_H \underline{W}$ is a trivial G-vector bundle because $G \times_H (\delta \times W) \cong (G \times_H W) \times \delta \cong G/H \times W \times \delta \cong (G/H \times \delta) \times W$. Let $G\Psi = G \times_H \Psi : G \times_H \underline{W} \to \xi'$. Since $G \times_H \Psi$ is a surjective semialgebraic G-morphism, ξ' is a strongly semialgebraic G-vector bundle. Thus we have obtained global semialgebraic sections $\bar{s}_1, \dots, \bar{s}_k, \dots, \bar{s}_l, \dots, \bar{s}_m$ over $G/H \times \delta$ such that they generate a finite dimensional G-invariant subspace of $\Gamma(\xi')$, for any $x \in G/H \times \delta$, $\bar{s}_1(x), \dots, \bar{s}_m(x)$ generate the fiber over x, $\bar{s}_i|_{G/H\times\partial\delta} = s_i|_{G/H\times\partial\delta}$ for $1 \leqslant i \leqslant k$ and that $\bar{s}_j$ vanish on $G/H \times \partial\delta$ for $k+1 \leqslant j \leqslant m$. Hence they can be defined over $B^{(n-1)} \cup G\overline{\sigma}$. This completes the proof. $\square$

4. Proof of Theorem 1.2

Now we first prove that κ defined in Introduction is surjective.

The following proposition is the equivariant homotopy property of G-vector bundles.

Proposition 4.1 [2,10]. *Let G be a compact Lie group. Let X be a paracompact G-space and let ξ be a topological G-vector bundle over a G-space Y. If $f, h : X \to Y$ are G-homotopic continuous G-maps, then $f^*(\xi)$ and $h^*(\xi)$ are topologically G-vector bundle isomorphic over X.*

The following proposition is a well-known fact on topological G-vector bundles.

Proposition 4.2 [1,14]. *Let G be a compact topological group and let X be a compact G-space. If ξ is a topological G-vector bundle over X, then there exist a G-representation space Ω and a continuous G-map $f : X \to G(\Omega, k)$ such that ξ is topologically G-isomorphic to $f^*(\gamma(\Omega, k))$.*

By Proposition 2.5, we know that every semialgebraic G-set M admits a semialgebraic G-strong deformation retraction to a compact semialgebraic G-subset A of M. Let ξ be a G-vector bundle over a semialgebraic G-set M. One can find a G-representation space Ω and a continuous G-map $f : A \to G(\Omega, k)$ such that $\xi|_A$ is topologically G-isomorphic to $f^*(\gamma(\Omega, k))$ by Proposition 4.2. Moreover $\xi \cong_G^{\mathrm{top}} r^*(\xi|_A) \cong_G^{\mathrm{top}} r^*(f^*(\gamma(\Omega, k))) \cong_G^{\mathrm{top}} (f \circ r)^*(\gamma(\Omega, k))$. Therefore there is a continuous G-map $\varphi := f \circ r : X \to G(\Omega, k)$ such that $\xi \cong_G^{\mathrm{top}} \varphi^*(\gamma(\Omega, k))$.

Recall that every compact semialgebraic group has a compact Lie group structure. Thus we have the following corollary.

Corollary 4.3. *Let G be a compact semialgebraic group. If ξ is a G-vector bundle over a semialgebraic G-set M, then there exist a G-representation space Ω and a continuous G-map $f : M \to G(\Omega, k)$ such that ξ is topologically G-isomorphic to $f^*(\gamma(\Omega, k))$.*

Corollary 4.3 states that every topological G-vector bundle over a semialgebraic G-set has a continuous classifying G-map.

By Corollary 4.3, Proposition 2.5 and Proposition 4.1, we have the following corollary.

Corollary 4.4. *Let G be a compact semialgebraic group and let M be a semialgebraic G-set. Then every topological G-vector bundle over M is topologically G-isomorphic to a semialgebraic G-vector bundle.*

Therefore we obtain the surjectivity of κ.

Next we prove that κ is injective. Before proving the general case, we show it when the base space is compact.

Proposition 4.5. *Let G be a compact semialgebraic group and let ξ be a semialgebraic G-vector bundle over a compact semialgebraic G-set M. Then every continuous G-section of ξ can be approximated by a semialgebraic G-section.*

Proof. By Theorem 3.5, ξ is strongly semialgebraic. Hence one can take a semialgebraic G-map $f: M \to G(\Omega, k)$ such that ξ is semialgebraically G-isomorphic to $f^*(\gamma(\Omega, k))$ for some G-representation space Ω, where k denotes the rank of ξ. Thus we can identify ξ with a subbundle of the trivial G-vector bundle $\underline{\Omega} = M \times \Omega$. Under this identification, a map $h: M \to \Omega$ is a section of ξ if and only if $f(x)h(x) = h(x)$ for any $x \in M$. Let l be a continuous G-section of ξ. We regard l as a continuous G-map M to Ω. We approximate l by a polynomial map p. After averaging p, we may assume that p is an equivariant polynomial map by [7, 4.1]. Put $s(x) = f(x)p(x)$. Then we have $f(x)s(x) = f(x)^2 p(x) = f(x)p(x) = s(x)$ for any $x \in M$ because $f(x) \in G(\Omega, k)$ for any $x \in M$. Therefore $s(x)$ is the required semialgebraic G-section. □

Theorem 4.6. *Let G be a compact semialgebraic group. Let ξ and η be semialgebraic G-vector bundles over a compact semialgebraic G-set. If $\xi \cong_G^{\mathrm{top}} \eta$, then $\xi \cong_G^{\mathrm{sem}} \eta$.*

Proof. By Proposition 3.3, $\mathrm{Hom}(\xi, \eta)$ is also semialgebraic. Take a topological G-isomorphism f between ξ and η. We can see f as a continuous G-section of $\mathrm{Hom}(\xi, \eta)$ which lies in $\mathrm{Iso}(\xi, \eta)$. By Proposition 4.5, f can be approximated by a semialgebraic G-section s of $\mathrm{Hom}(\xi, \eta)$. If this approximation is sufficiently close, then s gives the desired semialgebraic G-isomorphism because $\mathrm{Iso}(\xi, \eta)$ is open in $\mathrm{Hom}(\xi, \eta)$. □

Corollary 4.7. *Let G be a compact semialgebraic group. If M is a compact semialgebraic G-set, then the canonical map $\kappa: \mathrm{Vect}_G^{\mathrm{sem}}(M) \to \mathrm{Vect}_G^{\mathrm{top}}(M)$ is bijective.*

Now we prove Theorem 1.2 when the base space is general.

Let M be a semialgebraic G-set. By Proposition 2.5, one can find a compact semialgebraic G-subspace A of M and a semialgebraic G-retraction $r : M \to A$. Let $i : A \to M$ denote the inclusion. Then we have the following lemma.

Lemma 4.8.

(1) *The map* $r^* : \mathrm{Vect}_G^{\mathrm{sem}}(A) \to \mathrm{Vect}_G^{\mathrm{sem}}(M)$ *is injective.*

(2) *The map* $i^* : \mathrm{Vect}_G^{\mathrm{sem}}(M) \to \mathrm{Vect}_G^{\mathrm{sem}}(A)$ $(\xi \mapsto \xi|_A)$ *is surjective.*

Proof. Consider the composition map $\mathrm{id}_A = r \circ i : A \to M \to A$. $\square$

In the topological category, the map $r^* : \mathrm{Vect}_G^{\mathrm{top}}(A) \to \mathrm{Vect}_G^{\mathrm{top}}(M)$ is bijective. In semialgebraic category, we have the following.

Proposition 4.9. $r^*(\xi|_A) \cong_G^{\mathrm{sem}} \xi$ *for all* $\xi \in \mathrm{Vect}_G^{\mathrm{sem}}(M)$.

Proof. We use the same notation as in the proof of Proposition 2.5. Let ξ be a semialgebraic G-vector bundle over M. For each open G-n-cell $G\sigma \in \mathcal{C}_n$ such that $G\sigma \cap A = \emptyset$, there exists a semialgebraic characteristic G-map $Gf_\sigma : G/H \times \delta \to G\overline{\sigma} \subset M$.

By induction on n, suppose that we have a semialgebraic G-isomorphism

$$\Phi_{n-1} : \xi|_{X_{n-1}} \to (r|_{X_{n-1}})^*(\xi|_A).$$

Now we construct $\Phi_n : \xi|_{X_n} \to (r|_{X_n})^*(\xi|_A)$.

Fix a G-cell $G\sigma \in \mathcal{C}_n$ and let $Gf_\sigma : G/H \times \delta \to G\overline{\sigma} \subset M$ be its characteristic G-map. Note that the H-action on δ is trivial. Let ξ' and ξ'' denote $(Gf_\sigma)^*(\xi)$ and $\xi'|_\delta = (f_\sigma)^*(\xi)$, respectively, where f_σ denotes the restriction of Gf_σ to $\{eH\} \times \delta = \delta$. Let $\varphi : \delta \to \mathbb{G}(\Omega, k)^H \subset \mathbb{G}(\Omega, k)$ be a classifying H-map corresponding to ξ''. Since both Δ and $\mathbb{G}(\Omega, k)$ are compact, applying local triviality of semialgebraic maps (e.g., [3, 9.3.2]), there exists a semialgebraic H-extension $\varphi' : \Delta \to \mathbb{G}(\Omega, k)$ of φ. Using φ', we get a semialgebraic H-vector bundle $(\varphi')^*(\gamma(\Omega, k))$ over Δ such that $(\varphi')^*(\gamma(\Omega, k))|_\delta$ is semialgebraically H-isomorphic to ξ''. Let r_Δ be the H-deformation retract Δ to $\partial\delta = \delta - \mathrm{int}(\delta)$ and r_δ its restriction on δ. Since Δ is compact H-contractible semialgebraic set, we know by Theorem 4.6 that the pull-back bundle $(\varphi')^*(\gamma(\Omega, k))$ is semialgebraically H-isomorphic to a trivial semialgebraic H-vector bundle $\Delta \times V$ for some H-representation space V. Since ξ'' is isomorphic to $r_\delta^*((\varphi')^*(\gamma(\Omega, k))|_{\partial\delta})$ and $r_\delta^*(\xi''|_{\partial\delta})$ is isomorphic to $r_\delta^*(((\varphi')^*(\gamma(\Omega, k))|_{\partial\delta})$, both are trivial vector bundles, and they are isomorphic. However, this isomorphism between ξ and $r_\delta^*(\xi''|_{\partial\delta})$ may not be the identity on $\partial\delta$, thus we need more argument.

Identify ξ'' with $\delta \times V$ and $\xi''|_{\partial\delta}$ with $\partial\delta \times V$. Let the isomorphism on $\partial\delta$ denote $l(x, v) = (x, l_x(v))$. Then l_x is an H-linear map for all $x \in \partial\delta$. Now we define $\Psi'_\delta : \delta \times V \to \delta \times V$, $\Psi'_\delta(x, v) = (x, l_{r_\delta(x)}(v))$. Then its induced map $\Psi_\delta : \xi \to r_\delta^*(\xi''|_{\partial\delta})$ is identity on $\partial\delta$.

We define a semialgebraic G-isomorphism

$$G\Psi_\delta : \xi' \to (G \times_H r_\delta)^*\big(\xi'|_{G/H\times\delta}\big)$$

by $G \times_H \Psi_\delta$. Since $G\Psi_\delta$ is identity on $G/H \times \partial\delta$, it induces a semialgebraic G-isomorphism $\Psi_{G\sigma} : \xi|_{G\sigma} \to (r_{G\sigma})^*(\xi|_{G\partial\sigma})$, hence it provides $\Psi^n : \xi|_{X_n} \to (r_n)^*(\xi|_{X_{n-1}})$. Now we get

$$\Phi_n : \xi|_{X_n} \to (r|_{X_n})^*(\xi|_A)$$

by the composition $\Psi_1^{n-1} \circ \Psi^n$, where $\Psi_1^{n-1} : (r_n)^*(\xi|_{X_{n-1}}) \to (r_n)^*(r|_{X_{n-1}})^*(\xi|_A)$ is the induced semialgebraic G-isomorphism from Φ_{n-1}. □

The following corollary is a consequence of Proposition 4.9.

Corollary 4.10. *The map $r^* : \mathrm{Vect}_G^{\mathrm{sem}}(A) \to \mathrm{Vect}_G^{\mathrm{sem}}(M)$ is bijective.*

Now we prove injectivity of κ as follows.

Proof of injectivity of κ. The induced maps

$$r^* : \mathrm{Vect}_G^{\mathrm{sem}}(A) \to \mathrm{Vect}_G^{\mathrm{sem}}(M),$$
$$r^* : \mathrm{Vect}_G^{\mathrm{top}}(A) \to \mathrm{Vect}_G^{\mathrm{top}}(M)$$

by $r : A \to M$ are bijective by Corollary 4.10. Let

$$\kappa_A : \mathrm{Vect}_G^{\mathrm{sem}}(A) \to \mathrm{Vect}_G^{\mathrm{top}}(A), \quad \kappa_A\big([\eta]_{\mathrm{sem}}\big) = [\eta]_{\mathrm{top}}.$$

Then two maps

$$\kappa \circ r^*, r^* \circ \kappa_A : \mathrm{Vect}_G^{\mathrm{sem}}(A) \to \mathrm{Vect}_G^{\mathrm{top}}(M)$$

coincide. Since A is compact and by Corollary 4.7, κ_A is bijective. Therefore κ is bijective. Therefore the proof of Theorem 1.2 is complete. □

Theorem 1.3 follows from Theorem 1.2 and Proposition 4.1.

5. Proof of Theorem 1.4

For general terminology of Nash category, we refer the reader to [3,8,16].

To prove Theorem 1.4, we need the Nash approximation theorem [15] as follows.

Theorem 5.1 [15]. *Let M and N be affine Nash manifolds and let $f : M \to N$ be a $\mathcal{C}^r$ Nash map, where $r < \infty$. Then f can be approximated by a $\mathcal{C}^\infty$ Nash map $h : M \to N$ in the $\mathcal{C}^r$ topology.*

In Theorem 5.1, we have to emphasize this $\mathcal{C}^r$ topology. It is defined in [15] and it has desirable properties [15,16]. Remark that the topology does not necessarily agree with the Whitney C^r topology in general.

Recall that the averaging function when G is a finite group.

Let G be a finite group and let f be a C^r Nash G-map from a C^r Nash G-manifold M to a G-representation Ω and $0 \leqslant r \leqslant \omega$. Then the averaging function $A(f): M \to \Omega$ is

$$A(f)(x) = (1/n)\sum_{i=1}^{n} g_i f\big(g_i^{-1}x\big),$$

where $G = \{g_1, \ldots, g_n\}$.

Hence $A(f)$ is a Nash G-map by definition, and A is a continuous self-map of the set of C^r Nash maps M to Ω with respect to the $\mathcal{C}^r$ topology if $0 \leqslant r < \infty$ by [16, II.1.5] and [16, II.1.6].

Proof of Theorem 1.4. By Theorem 1.2, they are semialgebraically G-isomorphic. Then this isomorphism defines a semialgebraic G-section f of $\mathrm{Iso}(\xi, \xi') \subset \mathrm{Hom}(\xi, \xi')$. Since $\mathrm{Hom}(\xi, \xi')$ is a strongly Nash G-vector bundle by [8, 3.1], we can take its classifying G-map $F: M \to G(\Xi, k)$ for some G-representation Ξ, where k denotes the rank of $\mathrm{Hom}(\xi, \xi')$. Thus we can identify a semialgebraic G-section (respectively a Nash G-section) H' of $\mathrm{Hom}(\xi, \xi')$ with a semialgebraic G-map (respectively a Nash G-map) H from M to Ξ such that $F(x)H(x) = H(x)$ for any $x \in M$.

By Theorem 5.1, one can find a Nash map h_1 approximating f in the C^0 topology. After applying the averaging process, we may assume that h_1 is a Nash G-map approximating f because A is a continuous self-map. Hence putting $h_2: M \to \Xi,\ h_2(x) = F(x)h_2(x)$, then h_2 gives a Nash G-section h of $\mathrm{Hom}(\xi, \xi')$ because $F(x)h_2(x) = F(x)^2 h_2(x) = F(x)h_2(x) = h_2(x)$ for any $x \in M$. If this approximation is sufficiently close, then h lies in $\mathrm{Iso}(\xi, \xi')$. Thus it gives a Nash G-isomorphism. □

We have the following corollary by Theorem 1.4 and Proposition 4.1.

Corollary 5.2. *Let G be a finite group and let ξ be a strongly Nash G-vector bundle over an affine Nash G-manifold N. If $f,\ h: M \to N$ are Nash G-maps between affine Nash G-manifolds which are G-homotopic, then the pull-back bundles $f^*(\xi)$ and $h^*(\xi)$ are Nash G-isomorphic.*

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